COUPLED THERMOELASTICITY OF BEAMS BASED ON THE FIRST-ORDER SHEAR DEFORMATION THEORY

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This paper presents the finite element solution of a beam based on the first-order shear deformation theory subjected simultaneously to arbitrary time-dependent thermal and uniform mechanical transverse shock loads. The beam is made of homogenous and isotropic materials. The equation of motion and the conventional coupled energy equation are simultaneously solved to obtain the displacement components and temperature distribution in the beam. Damping effect and internal friction are neglected in the beam. The governing partial differential equations of the problem are solved simultaneously using the Galerkin finite element method with C^1- continuous shape function leading to fast convergence of the solution. Results are presented for simply supported boundary conditions.

Keywords: *beam; first-order shear deformation theory; coupled thermoelasticity; Galerkin Method*

1 Introduction

The equations for a coupled, thermoelastic beam, including the effects of shear deformation and rotatory inertia, are derived by Jones [1]. Mcquillen and Brull [2] presented analytical solution for the dynamic, thermoelastic response of cylindrical shells using a variational theorem. Coupled thermally induced vibrations of Euler-Bernoulli and Timoshenko beam with one-dimensional heat conduction are investigated by Seibert and Rice [3]. Coupled thermoelasticity of beams is discussed by Massalas and Kalpakidis [4,5]. The analytical solution of the coupled thermoelasticity of beams with the Euler-Bernoulli assumption is given in [4], and that with Timoshenko assumption is given in [5]. In the treatment of these problems a linear approximation for temperature variation across the thickness direction of the beam is considered. Eslami and Vahedi [6] presented the one-dimensional coupled thermoelasticity problem of rods using the Galerkin finite element method. Manoach and Ribeiro developed a numerical procedure to study the coupled large amplitude thermoelastic vibrations of Timoshenko beams subjected to the thermal and mechanical loads using the finite difference approximation and modal coordinate transformations [7].

This paper presents the behavior of a beam under lateral thermal shock with coupled thermoelastic assumption. The analysis is based on the Galerkin finite element method, using a C^1- continuous shape function. The beam formulations are based on the first-order shear deformation theory.

2 Derivation of the Governing Equations

Consider a beam of rectangular cross section with height h and width b, as shown in Fig.1. Using the first-order shear deformation theory, the displacement components are

$$u(x,z,t)=u_0(x,t)-z\psi(x,t), \qquad w(x,t)=w_0(x,t). \tag{1}$$

where u is the axial displacement, w is the transverse displacement in the z direction and ψ is the rotation angle of the cross-section with respect to the longitudinal axis. The subscript zero denotes middle surface displacement. In terms of the displacement components, the normal and shear strains are given by

$$\varepsilon_x=u_{0,x}-z\psi_{,x}, \qquad \gamma_{xz}=w_{0,x}-\psi. \tag{2}$$

where a comma denotes partial differentiation.

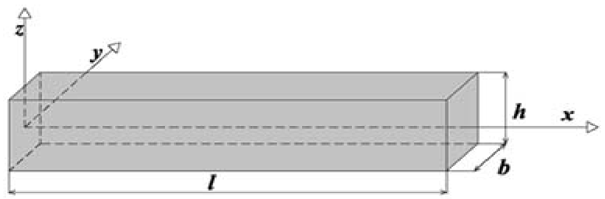

Figure 1: The beam and coordinates

Assuming that the beam material is linear elastic and isotropic, the stress-strain relations for the beam

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based on the assumed displacement components, including the shear deformation, are

$$\sigma_x = E(\varepsilon_x - \alpha\theta), \qquad \sigma_{xz} = k_s G \gamma_{xz}. \qquad (3)$$

where E is the modulus of elasticity, G is the shear modulus, k_s is the shear correction factor, α is the coefficient of thermal expansion, $\theta = T - T_0$ is the temperature change , and T_0 is the reference temperature, respectively.

The bending moment, the shear force, and the in-plane stress resultants per unit length are expressed by the stresses as follows:

$$M = b\int_z \sigma_x z\,dz, \qquad Q = b\int_z \sigma_{xz}\,dz, \qquad N = b\int_z \sigma_x\,dz. \qquad (4)$$

We assumed that the temperature change along the height direction is linear. This assumption is justified considering that the thickness of the beam is small with respect to its length [3,4]:

$$\theta = \theta_1(x,t) + \frac{z}{h}\,\theta_2(x,t). \qquad (5)$$

where θ_1 and θ_2 are unknown to be found through the solution of the coupled equations.

2.1 Equations of Motion

The equations of motion of a beam based on the first-order shear deformation theory is [6]:

$$\begin{gathered} N_{,x} = I_0 u_{0,tt} + I_1 \psi_{,tt}, \\ Q_{,x} + p(x,t) = I_0 w_{0,tt}, \\ M_{,x} - Q = I_1 u_{0,tt} - I_2 \psi_{,tt}. \end{gathered} \qquad (6)$$

where p is the applied surface lateral mechanical loading, $I_i = \int_z \rho z^i dz \ (i=0,1,2)$ is the mass moment of inertia and ρ is the mass density of the beam, respectively.

Substituting Eqs. (3), (4), and (5) into Eq. (6), the equations of motion become

$$\begin{gathered} hEu_{0,xx} - hE\alpha\theta_{1,x} = I_0 u_{0,tt}, \\ hk_s G(w_{0,xx} - \psi_{,x}) + p(x,t) = I_0 w_{0,tt}, \\ h^3 E\psi_{,xx} + h^2 E\alpha\theta_{2,x} + hk_s G(w_{0,x} - \psi) = 12 I_2 \psi_{,tt}. \end{gathered} \qquad (7)$$

2.2 Energy Equation

The first law of thermodynamics for heat conduction in beam in the coupled form is

$$(k\theta_{,i})_{,i} - \rho c_v \theta_{,t} - \alpha(3\lambda + 2\mu)T_0(\varepsilon_{ii})_{,t} = 0. \qquad i=1,2,3 \qquad (8)$$

where k, c_v, α, and ε_{ii} are the thermal conductivity, specific heat, coefficient of linear thermal expansion, and normal strain, respectively, and λ and μ are the Lamé constants. The energy equation for the beam based on first-order shear deformation theory is reduced to

$$Res = k\theta_{,xx} + k\theta_{,zz} - \rho c_v \theta_{,t} - \alpha E T_0(u_{0,xt} - z\psi_{,xt}) = 0. \qquad (9)$$

Using Eq. (5) and multiplying Eq. (9) by dz and zdz, integrating over height h, the residue Res of the energy equation may be made orthogonal with respect to dz and zdz, to provide two independent equations for two independent functions θ_1 and θ_2 as, [2]

$$hk\theta_{1,xx} - h\rho c_v \theta_{1,t} - hE\alpha T_0 u_{0,xt} + q^+ - q^- = 0. \qquad (10)$$

$$h^2 k\theta_{2,xx} - 12k\theta_2 - h^2\rho c_v \theta_{2,t} + h^3 E\alpha T_0 \psi_{,xt} + 6h(q^+ + q^-) = 0 \qquad (11)$$

The above system of energy equations are obtained assuming that $q^+(t)$ and $q^-(t)$ are the applied heat flux on the upper surface and lower surface of the beam, respectively.

3 Solution Procedure

To solve the simultaneous governing equations, dimensionless values are defined as

$$\begin{gathered} \bar{u}_0 = \lambda_1 \frac{u_0}{l}, \qquad \bar{\Psi} = \lambda_2 \psi, \\ \bar{w}_0 = \lambda_3 \frac{w_0}{l}, \qquad \bar{x} = \lambda_4 \frac{x}{l}, \\ \bar{t} = \lambda_5 \frac{c}{l} t, \qquad c^2 = \frac{E}{\rho}, \qquad \bar{\theta} = \lambda_6 \frac{\theta}{T_0}. \end{gathered} \qquad (12)$$

where l is length of the beam. The sign $(\bar{\ })$ indicates dimensionless value. The parameters λ_i are dimensionless coefficients introduced enabling us to change and balance the members of matrices in the FEM leading to convergence of the solution. In finite element method this technique may be used to make the members of the matrices of relatively the same order of magnitude.

3.1 Laplace Transform

The system of coupled equations is functions of the space variable x and time t. Traditional finite element solution for such problem is the time marching method. The solution presented in this paper is obtained by transfinite element method, where time is eliminated using the Laplace transform.

Using the dimensionless parameters and applying the Laplace transform to Eqs. (7), (10), and (11), give

$$\begin{gathered} A_1 \bar{U}_{0,\bar{x}\bar{x}} + A_2 \bar{\Theta}_{1,\bar{x}} + s^2 A_3 \bar{U}_0 = 0, \\ B_1 \bar{W}_{0,\bar{x}\bar{x}} + B_2 \bar{\Psi}_{,\bar{x}} + s^2 B_3 \bar{W}_0 + B_4 P = 0, \\ C_1 \bar{W}_{0,\bar{x}} + C_2 \bar{\Psi} + C_3 \bar{\Psi}_{,\bar{x}\bar{x}} + C_4 \bar{\Theta}_{2,\bar{x}} + s^2 C_5 \bar{\Psi} = 0, \\ D_1 \bar{\Theta}_{1,\bar{x}\bar{x}} + sD_2 \bar{\Theta}_1 + sD_3 \bar{U}_{0,\bar{x}} + D_4 Q^+ + D_5 Q^- = 0, \\ E_1 \bar{\Theta}_{2,\bar{x}\bar{x}} + E_2 \bar{\Theta}_2 + sE_3 \bar{\Theta}_2 + sE_4 \bar{\Psi}_{,\bar{x}} + E_5 Q^+ + E_6 Q^- = 0. \end{gathered} \qquad (13)$$

where s is the Laplace transform parameter, A's, B's, C's, D's, and E's are constants of coupled equations and

$$\overline{U}_0 = L[\overline{u}_0], \qquad \overline{\Psi} = L[\overline{\psi}], \qquad \overline{W}_0 = L[\overline{w}_0],$$
$$\overline{\Theta}_i = L[\overline{\theta}_i],\ i = 1,2 \qquad P = L[p], \qquad Q = L[q]. \tag{14}$$

where L is the Laplace transform operator.

3.2 Finite Element Modeling

The coupled system of Eqs. (13) may be solved by the Galerkin finite element method. The beam is divided into a number of straight elements. The base element *(e)* along the length of the beam is considered. The unknown functions of coupled equations, for instance W_0, in *(e)* may be approximated with the third order interpolation function $<N>$ as

$$W_0 = <W_{0p}> \{N_p\}. \quad p=1,2,3,4 \tag{15}$$

Considering a C^1- continuous shape function, the degrees of freedom for the element become

$$<W_{0p}> = < W_{0i} \quad W_{0,x|i} \quad W_{0j} \quad W_{0,x|j} >. \tag{16}$$

The assumed shape function insures the continuity of the nodal degrees of freedom, as well as their first derivative with respect to the variable x. Now, the formal Galerkin method may be applied to the system of Eqs. (13). The final finite element equation of motion, after assembling the matrix equations of each individual element, is obtained as

$$([M]s^2+[C]s+[K])\{X\}=\{F\}. \tag{17}$$

where $[M]$, $[C]$, $[K]$, $\{F\}$, and $\{X\}$ are mass, capacitance, stiffness , force, and unknowns matrices, respectively. The solution of Eq. (17) for the unknown matrix is obtained in terms of the Laplace parameters s. To obtain the solution in real time domain, the inverse Laplace transform must be carried out. This may be done by the method proposed by Durbine [8].

4 Results

To study of lateral thermal shock effect on the beam with coupled thermoelastic assumption, an aluminum beam of length 0.25m and height 0.0022m with simply supported boundary conditions is assumed. The material properties of aluminum are shown in Table 1. The thermal boundary conditions at the ends of the beam are assumed to be ambient temperature $T_0 = 298K$. The upper side of the beam is subjected to a unit Heaviside step function thermal shock while the lower side is insulated. The number of elements to obtain the results in this work is 50. The Galerkin finite element method, using the numerical inverse Laplace transforms, is used to obtain the solution.

The rapid convergence is the advantage of C^1-continuous shape function and the Galerkin finite element method.

Table 1: Material properties of aluminum

$E = 70\,\mathrm{G}pa$	$\rho = 2707\,\mathrm{kg/m^3}$
$\nu = 0.3$	$K = 204\, W/m^0K$
$\alpha = 23\times10^{-6}\ 1/^0\mathrm{K}$	$c_v = 903 J/kg^0K$

Fig.2 shows the temperature history of the middle surface of the beam in the midpoint. Due to classical coupled effect in the beam, the temperature peaks to a maximum value, and then oscillates about this value. The distribution of temperature of the middle surface of the beam is shown in Fig.3 for several times.

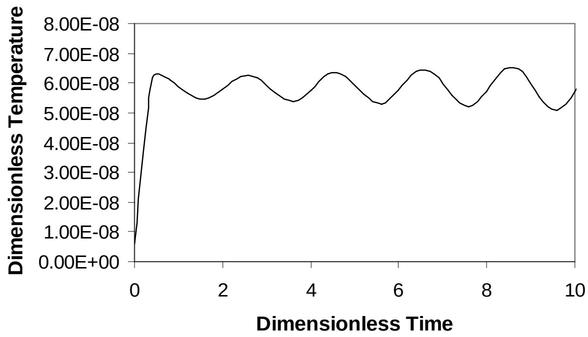

Figure 2: Temperature history for the midpoint of the beam for the middle surface

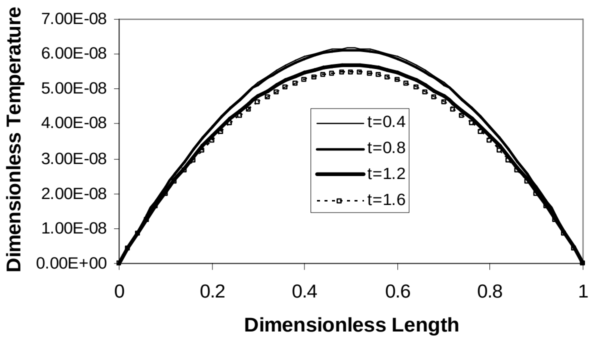

Figure 3: Temperature distribution for the middle surface of the beam

Fig.4 shows the deflection history of the middle length of the beam with respect to time. Due to the applied thermal shock, beam vibrated. Since flextural theory of thermoelasticity is used, no wave is revealed in deflection graph and therefore just the oscillations of the deflection are obtained. The symmetric distribution of deflection through the length of the beam is shown in Fig.5 for several times.

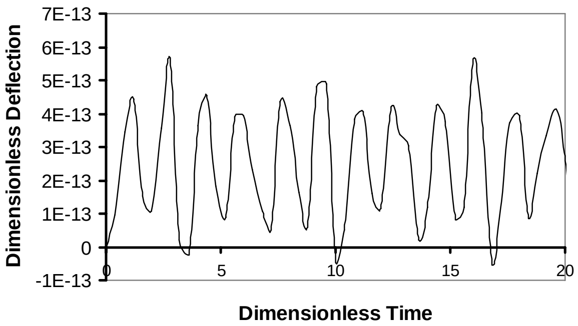

Figure 4: Deflection history of the midpoint of the beam

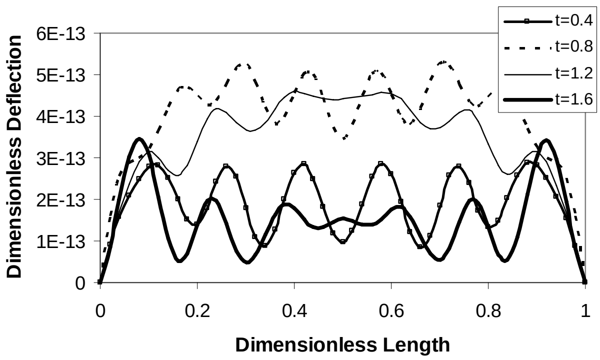

Figure 5: Distribution of deflection through the length of beam

Axial stress variations at the midpoint, top surface of the beam is shown in Fig.6. It may be found from this figure that due to the applying thermal shock to the beam, the axial stress has negative values, also the figure indicates an oscillation in the axial stress variation with small amplitude, resulting from the vibration of the beam and conversion between the thermal and mechanical energies.

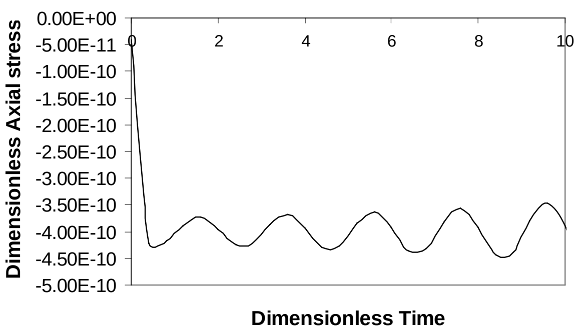

Figure 6: Axial stress history for the midpoint of the beam for the upper side

5 Conclusions

In the present paper, the coupled thermoelasticity of a beam based on the first-order shear deformation theory is investigated.

The beam is subjected to a thermal shock of unit step function on the upper side while lower side is insulated. Boundary conditions of the beam are taken to be simply supported and have temperature the same as the ambient. To solve the problem finite element Galerkin method with C1- continuous shape function is used. Moreover, to treat time dependency, Laplace transform technique is applied. The obtained results in the Laplace domain are returned to time domain using inverse Laplace transform.

Results show that because of coupled thermoelastic assumption in the beam, the temperature peaks to a maximum value, and then oscillates about this value. Since flextural theory of thermoelasticity is used, the stress wave is not observed across the beam thickness. therefore just the oscillations of the axial stress and deflection are obtained.

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